Domácí úkol 7 – určitý integrál

1) a) $\int_2^{1+\sqrt{3}} \frac{1}{x^2-2x+2}dx = \int_2^{1+\sqrt{3}} \frac{1}{(x-1)^2+1}dx = \left[\operatorname{arctg}(x-1)\right]_2^{1+\sqrt{3}} =$

$= \operatorname{arctg}\sqrt{3} - \operatorname{arctg} 1 = \frac{\pi}{3} - \frac{\pi}{4} = \underline{\underline{\frac{\pi}{12}}}$

( integrál Newtonův i Riemannův ((R) i (N)) – integrujeme fci spojitou v $\langle 2, 1+\sqrt{3}\rangle$ )

b) $\int_2^3 \frac{1}{1-x^2}dx = \frac{1}{2}\int_2^3 \left(\frac{1}{1-x} + \frac{1}{1+x}\right)dx = \frac{1}{2}\left[\ln|1+x| - \ln|1-x|\right]_2^3 =$

$= \frac{1}{2}\left[\ln\left|\frac{1+x}{1-x}\right|\right]_2^3 = \frac{1}{2}(\ln 2 - \ln 3) = \underline{\frac{1}{2}\ln\frac{2}{3}}$

( (R) i (N) – opět – integrál ze spojité fce v $\langle 2,3\rangle$ )

c) $\int_1^e x\ln^2 x\, dx$ ((R) i (N)) $\underset{pp}{=} \left| \begin{array}{ll} u' = x, & u = \frac{x^2}{2} \\ v = \ln^2 x, & v' = 2\ln x \cdot \frac{1}{x} \end{array} \right| = \left[\frac{x^2}{2}\ln^2 x\right]_1^e - \int_1^e x\ln x\, dx =$

$= \left| \begin{array}{ll} u' = x, & u = \frac{x^2}{2} \\ v = \ln x, & v' = \frac{1}{x} \end{array} \right| = \left[\frac{x^2}{2}\ln^2 x\right]_1^e - \left(\left[\frac{x^2}{2}\ln x\right]_1^e - \frac{1}{2}\int_1^e x\, dx\right) =$

$= \left[\frac{x^2}{2}\ln^2 x - \frac{x^2}{2}\ln x + \frac{x^2}{4}\right]_1^e = \frac{e^2}{2} - \frac{e^2}{2} + \frac{e^2}{4} - \frac{1}{4} = \underline{\frac{1}{4}(e^2-1)}$

d) $\int_0^1 \arcsin x\, dx = \left| \begin{array}{ll} u' = 1, & u = x \\ v = \arcsin x, & v' = \frac{1}{\sqrt{1-x^2}} \end{array} \right| = \left[x\arcsin x\right]_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}dx =$ (zde jen (N))

$= \left[x\arcsin x + \sqrt{1-x^2}\right]_0^1 = \frac{\pi}{2} - 1$

nebo substituce a potom integrace per partes

$\int_0^1 \arcsin x\, dx = \left| \begin{array}{l} \arcsin x = t \\ x = \sin t \\ t \in \langle 0, \frac{\pi}{2}\rangle \\ dx = \cos t\, dt \end{array} \right| = \int_0^{\frac{\pi}{2}} t\cos t\, dt \underset{pp}{=} \left| \begin{array}{ll} u' = \cos t, & u = \sin t \\ v = t, & v' = 1 \end{array} \right|$

$= \left[t\sin t\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin t\, dt = \frac{\pi}{2} - \left[-\cos t\right]_0^{\frac{\pi}{2}} =$

$= \frac{\pi}{2} - 1$

e) $\displaystyle\int_0^3 \frac{x}{\sqrt{1+x}}\,dx =$ substituce (rozpis.) (2VS) $\left|\begin{array}{l} \sqrt{1+x} = t \\ x = t^2-1 \\ dx = 2t\,dt \\ x=0 \to t=1 \\ x=3 \to t=2 \end{array}\right| = \displaystyle\int_1^2 \frac{t^2-1}{t}\cdot 2t\,dt =$

(R) i (N)

$$= 2\int_1^2 (t^2-1)\,dt = 2\left[\frac{t^3}{3} - t\right]_1^2 = 2\left(\frac{8}{3} - 2 - \left(\frac{1}{3} - 1\right)\right) = \frac{8}{3}$$

f) $\displaystyle\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx =$ (1VS) $\left|\begin{array}{l} 1-x^2 = t \\ -2x\,dx = dt \\ x=0 \to t=1 \\ x=1 \to t=0 \end{array}\right| = -\displaystyle\int_1^0 \frac{1}{2\sqrt{t}}\,dt = \int_0^1 \frac{1}{2\sqrt{t}}\,dt =$

(N)

$$= \left[\sqrt{t}\right]_0^1 = 1$$

(nebo hned i subst. $\left|\begin{array}{l} \sqrt{1-x^2} = t \\ \frac{1}{2\sqrt{1-x^2}}\cdot(-2x)\,dx = dt \\ x=0 \to t=1 \\ x=1 \to t=0 \end{array}\right|$ , pak

$$\int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx = \int_0^1 \frac{-1}{2\sqrt{1-x^2}}\cdot(-2x)\,dx = -\int_1^0 dt = \int_0^1 dt = \left[t\right]_0^1 = 1\ )$$

2*) $\displaystyle\int_0^\pi \frac{1}{1+3\cos^2 x}\,dx = \int_0^{\frac{\pi}{2}} \frac{1}{1+3\cos^2 x}\,dx + \int_{\frac{\pi}{2}}^{\pi} \frac{1}{1+3\cos^2 x}\,dx = 2\int_0^{\frac{\pi}{2}} \frac{1}{1+3\cos^2 x}\,dx$

((R) i (N))

2VS – $\operatorname{tg} x = t$ – platí ale $x \in \langle 0, \frac{\pi}{2})$ nebo $x \in (\frac{\pi}{2}, \pi\rangle$ !! – viz (*)

kde $\displaystyle\int_{\frac{\pi}{2}}^{\pi} \frac{1}{1+3\cos^2 x}\,dx =$ subst. $\left|\begin{array}{l} t = \pi - x \\ dx = -dt \\ x=\frac{\pi}{2} \to t=\frac{\pi}{2} \\ x=\pi \to t=0 \end{array}\right| = -\displaystyle\int_{\frac{\pi}{2}}^{0} \frac{1}{1+3\cos^2(\pi-t)}\,dt =$

$(\cos^2(\pi-t) = \cos^2 t)$ $= \displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+3\cos^2 t}\,dt$ , tedy stejný integrál $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+3\cos^2 x}\,dx$

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+3\cos^2 x}\,dx = \left| \begin{array}{l} \operatorname{tg} x = t \\ x = \operatorname{arctg} t \\ dx = \frac{1}{1+t^2}\,dt \\ \cos^2 x = \frac{1}{1+t^2} \\ x = 0 \to t = 0 \\ x = \frac{\pi}{2} \to t \to \infty \end{array} \right| = \int_0^{\infty} \frac{1}{1+\frac{3}{1+t^2}} \cdot \frac{1}{1+t^2}\,dt =$$

(R) i (N) (N)

$$= \int_0^{\infty} \frac{1}{4+t^2}\,dt = \frac{1}{4}\int_0^{\infty} \frac{1}{1+\left(\frac{t}{2}\right)^2}\,dt = \frac{1}{2}\left[\operatorname{arctg}\left(\frac{t}{2}\right)\right]_0^{\infty} = \underline{\underline{\frac{\pi}{4}}},$$

tedy,
$$\int_0^{\pi} \frac{1}{1+3\cos^2 x}\,dx = 2\int_0^{\frac{\pi}{2}} \frac{1}{1+3\cos^2 x}\,dx = \underline{\underline{\frac{\pi}{2}}}$$

<u>aplikace</u>

1. $$\underline{S(\omega) = \int_0^1 e^{\sqrt{x}}\,dx} \underset{2VS}{=} \left| \begin{array}{l} \sqrt{x} = t \\ x = t^2 \\ dx = 2t\,dt \\ x=0 \to t=0 \\ x=1 \to t=1 \end{array} \right| = 2\int_0^1 t e^t\,dt \underset{PP}{=}$$

$$= 2\left[te^t - e^t\right]_0^1 = 2$$

2. $$\underline{S(\omega) = \int_0^1 (x - \operatorname{arctg} x)\,dx} = \left[\frac{x^2}{2}\right]_0^1 - \int_0^1 \operatorname{arctg} x\,dx \underset{PP}{=} \left| \begin{array}{l} u' = 1,\ u = x \\ v = \operatorname{arctg} x,\ v' = \frac{1}{1+x^2} \end{array} \right|$$

$$= \left[\frac{x^2}{2}\right]_0^1 - \left( \left[x \operatorname{arctg} x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx \right) = \left[\frac{x^2}{2} - x\operatorname{arctg} x + \frac{1}{2}\ln(1+x^2)\right]_0^1$$

$$\cdots = \frac{1}{2}\left(1 + \ln 2\right) - \frac{\pi}{4}$$

3. $V = \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 x \, dx = 2\pi \int_0^{\frac{\pi}{2}} \cos^2 x \, dx = 2\pi \int_0^{\frac{\pi}{2}} \frac{1+\cos 2x}{2} \, dx =$

(integrál ze sudé fce)

$$= \pi \left[ x + \frac{\sin 2x}{2} \right]_0^{\frac{\pi}{2}} = \pi \cdot \frac{\pi}{2} = \frac{\pi^2}{2}$$

(nebo lze $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 x \, dx$ „spočítat" metodou integrace per partes)

a navíc:

(i) $f \in R(-a,a),\ a>0,\ f$ lichá $\Rightarrow \int_{-a}^{a} f(x)\,dx = 0$

Dk: $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = -\int_{0}^{a} f(t)\,dt + \int_{0}^{a} f(x)\,dx = 0$

(aditivita) (↑ na označení proměnné zde „nezáleží")

a $\int_{-a}^{0} f(x)\,dx = $ subst. $\left| \begin{array}{l} x=-t \\ dx=-dt \\ x=-a \to t=a \\ x=0 \to t=0 \end{array} \right| = -\int_{a}^{0} f(-t)\,dt = \int_{0}^{a} f(-t)\,dt = -\int_{0}^{a} f(t)\,dt$

ale $f(-t) = -f(t)$

analogicky

(ii) $f \in R(-a,a),\ a>0,\ f$ sudá (tj. $f(x)=f(-x)$) $\Rightarrow \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

a $\int_{-a}^{0} f(x)\,dx = -\int_{a}^{0} f(-t)\,dt = \int_{0}^{a} f(t)\,dt$

$f(-t) = f(t)$